On the critical temperature of the two-dimensional Ising spin glass models

1982 J. Phys. A: Math. Gen. 15 L133

(http://iopscience.iop.org/0305-4470/15/3/010)

J. Phys. A: Math. Gen. **15** (1982) L133–L137. Printed in Great Britain

LETTER TO THE EDITOR

On the critical temperature of the two-dimensional Ising spin glass models†

Lech Longa‡

Solid State Physics Laboratory, University of Groningen, Melkweg 1, 9718 EP Groningen, The Netherlands

Received 23 November 1981

Abstract. The critical temperature for a quenched Ising model on the square lattice, with vertical random interactions, is found exactly. The detailed phase diagram is obtained for the bond-dilute model and for the model with mixed ferro- and antiferromagnetic δ distribution of the bonds. For the class of the models with symmetric distribution of the bonds ($P(J) = P(-J)$) it is shown that the critical temperature is always equal to zero. An immediate consequence of this is the theorem about the absence of the second-order phase transitions for the square, fully random, quenched Ising model with symmetric distribution of the bonds.

Much work has recently been devoted to the study of static and dynamic properties of spin glasses (Fisher 1977, Kinzel and Fisher 1977). The experimental data show that the properties of random magnets differ qualitatively from those of ordinary magnetic materials. For example, at a certain 'freezing' temperature T_f, the magnetic susceptibility has a sharp cusp in the zero-magnetic field (Arrolt 1965). This cusp is strongly rounded even by a small magnetic field. At T_f no singularity occurs in the specific heat (Wegner and Keesom 1976).

From a theoretical point of view, the random quenched magnets are very complicated systems, since the spin degrees of freedom interact with each neighbour through a random bond interaction. So, we must first solve such completely anisotropic models and then average the calculated free energy over the probability distribution of all bonds. This is the reason why the problem of phase transitions in spin glass models is still a very puzzling matter. The Monte Carlo calculations, the high-temperature series expansions and the real space renormalisation group used for random Ising models give only very inconclusive answers (Kinzel and Fisher 1978, Fish and Harris 1977, Binder 1979). To go further, a concept of relevant and irrelevant disorder has been introduced (Kirkpatrick 1977). It is well known that there is a class of models, commonly known as Mattis (flat) models (Mattis 1976), for which randomness is trivial and can be eliminated. Therefore, Toulouse (1977) has advanced the concept of frustration as a measure of relevant disorder and realised the existence of a local (gauge) symmetry of random magnets at the microscopic level. Quite generally, in frustrated systems, as compared with unfrustrated ones, there is a density of 'defects'

† The results of this Letter were presented at the International Conference on Disordered Systems and Localization, Rome, 13–15 May, 1981.

‡ Permanent address: Department of Statistical Physics, Institute of Physics, Reymonta 4, 39-059 Kraków, Poland.

0305-4470/82/030133+05$02.00

which cannot be eliminated by a local gauge transformation. For a plane random Ising model this situation appears for a particular square, when one of the bonds has a different sign from the other three. Thus, the ground state energy is always larger than in the pure system and the state is highly degenerated. The questions which immediately arise are the following. How do the frustration and disorder-induced frustration influence the thermodynamic behaviour of the system for $T \geqslant 0$? In particular, how does the ferromagnetic critical temperature change as a function of the concentration of antiferromagnetic impurities, which give the non-vanishing probability of frustration of elementary cells? How do the properties of the model depend on the parameters of the distribution function of the bonds? The purpose of this Letter is to give a partial answer to these questions. The critical temperature T_c as a function of the distribution function parameters is found exactly for the McCoy and Wu version of the spin glass model. Some exact results for quenched, fully random, Ising planar models are also presented.

The McCoy and Wu version of the spin glass model (MCWSG) is a kind of quenched, square Ising model, described by the Hamiltonian

$$H = -J_0 \sum_{j,k} \sigma_{j,k}\sigma_{j,k+1} - \sum_{j,k} J_k \sigma_{j,k}\sigma_{j+1,k} \tag{1}$$

where j labels the rows and k labels the columns of the lattice (figure 1). All vertical bonds J_k are independent random variables with a probability distribution function $P(J_k)$. This means that all the interactions in a column k are the same and are allowed to vary from column to column.

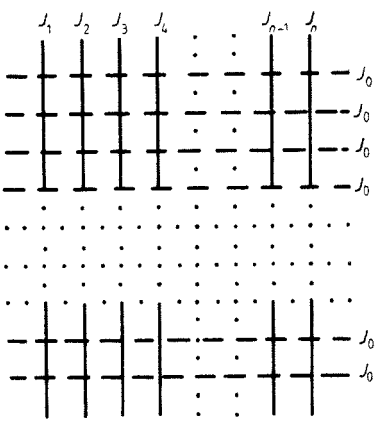

Figure 1. The MCWSG model.

McCoy and Wu (1968) have solved a similar model (which we will call here the MCW model), obtained by changing J_k into J_j in equation (1). The difference, however, between the MCWSG model and the MCW model is essential. For example, when $P(J_k)$ in the MCWSG model (or $P(J_j)$ in the MCW model) is equal to $(1-p)\,\delta(J_k - J_0) + p\delta(J_k + J_0)$ $(0 \leqslant p \leqslant 1)$, the MCWSG model yields a non-trivial probability of frustration of elementary cells. This situation appears when the vertical bonds on some elementary cells are of different signs. One can easily check that the probability of such a configuration of the bonds for an arbitrary, elementary cell for the MCWSG model is

equal to $2p(1-p)$. On the other hand the MCW model is equivalent to the Mattis (flat) model (Mattis 1976), the model in which there is no frustration.

Following the derivation of the free energy given by McCoy and Wu (1968), one can obtain the free energy and the critical temperature formulae for the model (1).

The free energy per particle F:

$$-\beta F = \ln(2\cosh\beta J_0) + \int_{-\infty}^{+\infty} P(J)\ln(\cosh\beta J)\,\mathrm{d}J$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_{-\infty}^{+\infty}\mathrm{d}J\int_{-\infty}^{+\infty}\mathrm{d}x\,\ln[a(\beta J,\theta)x + w^2(\beta J_0)]\nu(x,\beta J,\beta J_0,\theta)P(J)$$
$$+\frac{1}{4\pi}\int_{-\pi}^{\pi}\mathrm{d}\theta\int_{-\infty}^{+\infty}P(J)\ln|1+w(\beta J)\,\mathrm{e}^{\mathrm{i}\theta}|^2\,\mathrm{d}J, \tag{2}$$

where

$$a(\beta J,\theta) = \frac{2w(\beta J)\sin\theta}{|1+w(\beta J)\,\mathrm{e}^{\mathrm{i}\theta}|^2}, \qquad b(\beta J,\theta) = \frac{1-w^2(\beta J)}{|1+w(\beta J)\,\mathrm{e}^{\mathrm{i}\theta}|^2},$$
$$w(x) = \tanh(x),$$

and where $\nu(x)$ is the solution of the integral equation

$$\nu(x) = \int_{-\infty}^{+\infty}\mathrm{d}x'\int_{-\infty}^{+\infty}\mathrm{d}J\,\delta\left(x - \frac{(a^2(\beta J,\theta)+b^2(\beta J,\theta))x' + a(\beta J,\theta)w^2(\beta J_0)}{a(\beta J,\theta)x' + w^2(\beta J_0)}\right)P(J)\nu(x') \tag{3}$$

under the constraint

$$\int_{-\infty}^{+\infty}\nu(x)\,\mathrm{d}x = 1, \qquad \forall x\ \nu(x)\geqslant 0;$$

k_B is the Boltzmann constant.

Critical temperature:

$$w(J_0/k_BT_c) = \exp(-2|\bar{J}|/k_BT_c), \qquad \bar{J} = \int_{-\infty}^{+\infty}J_kP(J_k)\,\mathrm{d}J_k. \tag{4}$$

The absolute value of $\bar{J}$ in equation (4) is a consequence of the fact that the MCWSG model allows for negative values for the bonds. Thus the obtained condition must reflect the symmetry between ferro- and antiferromagnetic states.

For any given J_0 and $P(J_k)$, there exists only one non-negative critical temperature T_c that will satisfy equation (4). This is so, because condition (4) for $T = T_c$ is exactly the same as in the case of the pure Ising, square model with horizontal bonds equal to J_0 and vertical bonds equal to $|\bar{J}|$. All the remaining properties of formula (4) are dependent on the form of the probability distribution function $P(J_k)$.

Finally, we would like to point out that the critical temperature of the MCWSG model gives an upper bound for the critical temperature of the fully random model†.

† For the fully random model each bond is an independent random variable.

The following theorem is valid for the MCWSG model:

$$\forall P(J)\ \exists \tilde{J}_0(P)\colon \forall |J_0| \geqslant |\tilde{J}_0(P)|$$

$$T_c \text{ (fully random model; } P) \leqslant T_c(\text{MCWSG}, J_0; P). \tag{5}$$

It is easy to obtain the proof of this theorem by pointing out that in the limit of strong J_0 interaction, the two-particle correlation function, defined as

$$\overline{\lim_{l\to\infty} \lim_{J_0\to\infty} \langle \sigma_{j,k}\sigma_{j,k+l}\rangle},$$

approaches its limiting value $+1$ or -1 (here $\langle A\rangle$ denotes the thermodynamic average and $\bar{A}$ denotes the quenched average). From this it follows that our theorem is valid for distribution functions $P(J)$ which are non-zero on finite intervals of J.

As a consequence of inequality (5) and formula (4) the theorem about the absence of the second-order phase transition at temperatures greater than zero is valid for the square, fully random, quenched Ising model with symmetric distribution of the bonds ($P(J) = P(-J)$). A special case of symmetric $P(J)$ is the gaussian distribution function centred around $J = 0$ (Reed 1979).

Now we would like to discuss in detail the properties of critical lines for the bond-diluted model (BD) with

$$P(J_k) = (1-p)\delta(J_k) + p\delta(J_k - J_0) \tag{6}$$

and for the MCWSG frustration model (Longa 1980)

$$P(J_k) = (1-p)\delta(J_k + J_0) + p\delta(J_k - J_0), \qquad J_0 \geqslant 0. \tag{7}$$

In these cases equation (4) has the form

$$w(1/\tau_c) = \exp(-2p/\tau_c), \qquad \text{BD model,} \tag{8}$$

$$w(1/\tau_c) = \exp(-2|1-2p|\tau_c), \qquad \text{MSWSG model,} \tag{9}$$

where $\tau_c = k_B T_c/J_0$. The critical temperature τ_c, obtained from equations (8) and (9), plotted against the parameter p is illustrated in figure 2.

For the BD model the critical temperature decreases with decreasing probability p, for $0 \leqslant p \leqslant 1$. It reaches the limiting value 0 when the concentration of diluted bonds is equal to 1 ($p_c = 0$)†. This percolation threshold has a very nice geometrical interpretation. For $p = 0$, the BD model is equivalent to the set of non-interacting chains (broken lines in figure 1) and thus $\tau_c = 0$. For $p > 0$ all chains are spoiled by vertical interactions. This indicates that the long-range magnetic order is possible also for $\tau_c > 0$. In the neighbourhood of the point $p = 1$ the phase diagram $\tau_c(p)$ can be approximated by a straight line, with slope $s_{BD} = (\partial\tau_c/\partial p)_{p=1} = 1/\ln(1+\sqrt{2}) = 1.135$.

For the MCWSG model, the critical line is (as expected) symmetric with respect to the exchange of p and $1-p$. Additionally, one can easily check that the critical temperature decreases with decreasing probability p, for $0.5 \leqslant p \leqslant 1$. It reaches the limiting value 0 for $p = \frac{1}{2}$. The slope $s_{MCWSG} = (\partial\tau_c/\partial p)_{p=1} = 2/\ln(1+\sqrt{2}) = 2.269$.

The behaviour of the phase diagrams (8) and (9) is similar to this one known for fully random models (Tsallis and Levy 1980, Jayaprakash *et al* 1978, Guilmin and Turban 1980). Each critical temperature line increases from $\tau_c = 0$ at p_c to the pure

† p_c means the percolation threshold.

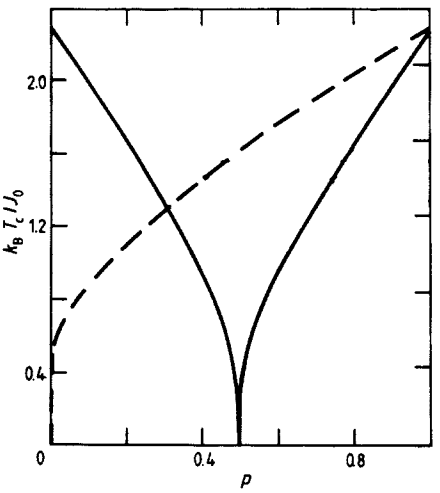

Figure 2. Critical temperature lines for the (*a*) bond-diluted model (broken line); (*b*) MCWSG model (full line).

Ising value $\tau_c = 2/\ln(1+\sqrt{2})$ at $p = 1$. For a wide range of values of p the critical temperature diagram can be approximated by a straight line. Close to p_c it decreases rapidly to zero.

Finally, we would like to add that the critical temperature is weakly dependent on the shape of the distribution function. Two different distributions with the same values of the first moment lead to indistinguishable phase boundaries.

The author is indebted to Professor A J Dekker and Dr W H de Jeu for kind hospitality during his stay at the University of Groningen. It is also a pleasure to thank Professor R Maynard, Drs M Gabay and I Morgenstern for valuable discussions.

References

Arrolt A 1965 *J. Appl. Phys.* **36** 1093
Binder K (ed) 1979 *Monte Carlo Methods in Statistical Physics* (Berlin: Springer) pp 301–33
Fish R and Harris A B 1977 *Phys. Rev. Lett.* **38** 785
Fisher K H 1977 *Physica* **86–88B** 813
Guilmin P and Turban L 1980 *J. Phys. C: Solid State Phys.* **13** 4077
Jayaprakash C, Riedel E J and Wortis M 1978 *Phys. Rev.* B **18** 2244
Kinzel W and Fisher K H 1977 *J. Phys. F: Metal Phys.* **7** 2163
—— 1978 *J. Phys. C: Solid State Phys.* **11** 2115
Kirkpatrick S 1977 *Phys. Rev.* B **16** 4630
Longa L 1980 *Physica* **103A** 633
Mattis D C 1976 *Phys. Lett.* **56A** 421
McCoy B M and Wu T T 1968 *Phys. Rev.* **176** 631
Reed 1979 *J. Phys. C: Solid State Phys.* **12** L799
Toulouse G 1977 *Commun. Phys.* **2** 115
Tsallis C and Levy S V F 1980 *J. Phys. C: Solid State Phys.* **13** 465
Wegner L E and Keesom P H 1976 *Phys. Rev.* B **13** 4053